# ON SOME PROPERTIES OF THE RANK-WEIGHTED MEANS

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Summary

Here is considered a class of measures of central tendency of frequency distributions, their uses illustrated, and a systematic account of their various properties, *viz.*, consistency, unbiasedness, existence of moments, asymptotic normality as well as the asymptotic efficiency, has been given. A consistent and distribution-free estimate of the variance of such a statistic has also been supplied.

## 1. Introduction

Let $X_1, \cdots, X_n$ be $n$ independent observations drawn at random from a population having a continuous cumulative distribution function (*cdf*) $F(x)$. Let $X_{(1)} < \cdots < X_{(n)}$ be the sample ordered variables. We then define a $k$-th order rank-weighted mean as

$$T_k(X_1, \cdots, X_n) = \binom{n}{2k+1}^{-1} \sum_{j=k+1}^{n-k} \binom{j-1}{k}\binom{n-j}{k} X_{(j)} \quad (1.1)$$

where $k$ is some non-negative integer. In the particular case: $k = 0$, $T_0$ reduces to the sample mean, and is known to have certain well-known properties. Again in the case: $k = [(n-1)/2]$, $[s]$ being the largest integer contained in $s$; $T_{[n-1/2]}$ reduces to the sample median. Now for some *cdf*'s, the sample mean does not appear to be very suitable (*e.g.*, the double exponential distribution or the Cauchy distribution), and further the sample mean is sensitive to variations of the sample outlying observations. Again, in most of the cases, the sample median is of comparatively low efficiency, as a measure of the population location parameter; and further it is not at all sensitive to minor fluctuations of the sample outlying observations. From this standpoint, it appears that an optimum measure of central tendency of a frequency distribution should fairly be dependent on all the sample observations, should not be materially affected by the minor fluctuations of the sample outlying observations, and further it should be reasonably efficient. The class of statistics $\{T_k\}$ appears to be quite suitable in this respect. Further the sample mean is not usable for censored or truncated samples. In such a case, the optimum measure of central tendency, as may be found by the method of maximum likelihood or the method of least squares on ordered variables, depends appreciably on the parent *cdf*, and further appears to be very cumbrous in respect of the computations involved and, the usual non-parametric approaches are generally of considerable low efficiency. Here also, we can use our class of statistics $\{T_k\}$, and the advantages are that these are computationally simpler, independent of the parent *cdf* $F(x)$, and have reasonably high efficiency for many common type of *cdf*'s.

In the particular case, the statistic $T_1$ has been proposed and studied by Ghosh (1950). The same statistic has also been considered by way of illustration, by the present author (Sen, 1960), in establishing certain convergence-properties of Hoeffding's (1948) U-statistic. The object of the present investigation is to give a systemaic account of the various properties of the class of statistics $\{T_k\}$, namely its consistency, unbiasedness, existence of its moments, asymptotic normality and asymptotic efficiency.

## 2. Some Moment-Properties of $\{T_k\}$

Here we propose to prove some theorems on the existence of the moments of $T_k$, and on the unbiasedness of $T_k$, for a class of parent *cdf*'s.

Theorem 2.1. For any continuous *cdf* $F(x)$ possessing a finite $\delta$-th order moment, for some $\delta > 0$, the $r$-th moment of $T_k$ in a sample of size $n$, exists for all real and positive $r$, satisfying

$$r \leqslant (k+1)\,\delta ; \quad n \geqslant 2k+1.$$

*Proof.*—Corresponding to the $2k+1$ independent observations $X_{\alpha_1}, \cdots, X_{\alpha_{2k+1}}$, we define the following counter-function

$$\phi_k(X_{\alpha_1}, \cdots, X_{\alpha_{2k+1}}) = X^*(\alpha_1, \cdots, \alpha_{2k+1}), \tag{2.1}$$

where $X^*(\alpha_1, \cdots, \alpha_{2k+1})$ is the median of $2k+1$ observations $X_{\alpha 1}, \cdots, X_{\alpha 2+1}$. Then obviously $X^*$ is a measurable function, and it is symmetric in its arguments. Now following Hoeffding (1948) we define the corresponding U-statistic as

$$U_k(X_1, \cdots, X_n) = \binom{n}{2k+1}^{-1} \sum_{C_k} \phi_k(X_{\alpha_1}, \cdots, X_{\alpha_{2k+1}}), \tag{2.2}$$

where the summation $C_k$ extends over all possible $1 \leqslant \alpha_1 < \cdots < \alpha_{2k+1} \leqslant n$. It is then readily proved that

$$T_k(X_1, \cdots, X_n) = U_k(X_1, \cdots, X_n) \tag{2.3}$$

It thus follows from (2.1) through (2.3) that a necessary and sufficient condition for the existence of the $r$-th moment of $T_k$ is that the $r$-th moment of $\phi_k$ exists, *i.e.*, the $r$-th moment of the sample median in a sample of size $(2k+1)$ exists. Also it follows from a theorem on the existence of the moments of the sample ordered variables, proved earlier by the author (Sen, 1959), that for any continuous *cdf* $F(x)$ possessing a finite $\delta$-th order moment, for some $\delta > 0$, the $r$-th moment of the sample median, in a sample of size $(2k+1)$ exists for all $r \leqslant (k+1)\delta$.

Hence, the theorem.

It is thus seen that the greater is the value of $k$, the more relaxed is the assumption regarding the existence of the moments of the parent *cdf* $F(x)$, in respect of the existence of the moments of $T_k$. This property is particularly useful when the range of the parent *cdf* is extended to infinity on either side, and we are not sure about the high order of terminal contact, and thus where moments of the parent *cdf* beyond a certain order may not exist; the Cauchy distribution naturally provide a very common example. For this *cdf*, $T_1$ has a finite mean but no finite variance, $T_2$ has both a finite mean and a finite variance, and in general $T_k$ has finite moments up to the $k$-th order.

Now the sample mean is an unbiased estimator of the population mean, even if $X_1, \cdots, X_n$ are not identically distributed, but all have a common mean. Here the unbiasedness of $T_k$ has been proved under quite general regularity conditions.

THEOREM 2.2. Let $X_i$ have the continuous *cdf* $F_i(x)$ for $i = 1, \cdots, n$. If $F_1(x), \cdots, F_n(x)$ are all symmetrical about the common location $\mu$, taken to be equal to zero, (but these *cdf*'s are otherwise quite arbitrary); then the distribution of $T_k$ is also symmetrical about $\mu = 0$. If, in addition, all these *cdf*'s possess a finite $\delta$-th moment, for some $\delta \geqslant 1/(k+1)$, then $T_k$ is an unbiased estimator of $\mu$.

*Proof.*—It follows from (1.1) that:

$$T_k(-X_1, \cdots, -X_n) = -T(X_1, \cdots, X_n) \tag{2.4}$$

for all points $X = (X_1, \cdots, X_n)$ in the sample space $S$. Thus, if $S_\lambda$ is the set of points in the sample space $S$, for which $T_k > \lambda$, then the set of points $C(S_{-\lambda})$ for which $T_k < -\lambda$ is obtained by changing the signs of the co-ordinates of the points of $S_\lambda$ *i.e.*, if $[X_1, \cdots, X_n$, $\epsilon S_\lambda$, then $(-X_1, \cdots, -X_n) \epsilon C(S_{-\lambda})$, where $C$ denotes the complement] Now, by virtue of the symmetry of $F_i(x)$ around $\mu = 0$, for $i = 1, \cdots, n$, we have

$$p(X_1, \cdots, X_n) = p(-X_1, \cdots, -X_n), \text{ for all } X \epsilon S; \tag{2.5}$$

where $p(X_1, \cdots, X_n)$ stands for the joint density function of the sample point $X$. Thus, we get directly from (2.4) and (2.5) that for any real $\lambda$.

$$P\{X \epsilon S_\lambda\} = 1 - P\{X \epsilon C(S_{-\lambda})\}.$$

Hence, the distribution of $T_k$ is symmetrical about $\mu = 0$.

It now follows from (2.3) and (2.4) that

$$E\{T_k(X_1, \cdots, X_n)\} = \binom{n}{2k+1}^{-1} \sum_{C_k} E\{\phi_k(X_{\alpha_1}, \cdots, X_{\alpha_{2k+1}})\} \tag{2.6}$$

where the summation $C_k$ extends over all possible $1 \leqslant \alpha_1 < \cdots < \alpha_{2k+1} \leqslant n$. By virtue of the symmetry of the distribution of $T_k$ around $\mu = 0$, it is thus sufficient to prove that $E\{\phi_k(X_{\alpha_1}, \cdots, X_{\alpha_{2k+1}})\}$ exists under the stated regularity conditions.

Now,

$$E\{\phi_k(X_{\alpha_1}, \cdots, X_{\alpha_{2k+1}})\}$$

$$= \sum_{S_k} \int_0^1 \lambda \prod_{j=1}^{k} F_{j_i}(\lambda) \prod_{j=k+2}^{2k+1} [1 - F_{j_i}(\lambda)]\, dF_{j_{k+1}}(\lambda), \tag{2.7}$$

where the summation $S_k$ extends over the possible $\{(2k+1)!/k!k!\}$ combinations of the $(2k+1)$ integers $(\alpha_1, \cdots, \alpha_{2k+1})$ into three subsets of sizes $k$, 1, and $k$ respectively. Now the existence of the $\delta$-th moment of $F_i(x)$ implies that [*cf.* Sen (1959)]

(*a*) $|x|^\delta F_i(x)$ is bounded for all $x \leqslant 0$ and it tends to zero a $x \to -\infty$,

(*b*) $|x|^\delta [1 - F_i(x)]$ is bounded for all $x \geqslant 0$ and it converges to zero as $x \to \infty$,

and

(*c*) $\int_0^1 |x|^\delta dF_i(x) < \infty$, for all $i = 1, \cdots, n$, Thus, for any $\delta \geqslant 1/(k+1)$,

(*i*) $|\lambda|^{1-\delta} \prod_{j=1}^{k} F_{j_i}(\lambda)$ is bounded for all $\lambda \leqslant 0$ and it converges to zero as $\lambda \to -\infty$, and

(*ii*) $|\lambda|^{1-\delta} \prod_{j=k+2}^{2k+1} [1 - F_{j_i}(\lambda)]$ is bounded for all $\lambda \geqslant 0$ and it tends to zero as $\lambda \to \infty$; and hence from (2.7) and (*c*), we get directly that

$$E\{|\phi_k(X_{\alpha_1}, \cdots, X_{\alpha_{2k+1}})|\} < \infty, \text{ for any } \delta \geqslant 1/(k+1).$$

Hence, the theorem.

*Mean and variance of $T_k$ when $X_1, \cdots, X_n$ are identically distributed*

If $X_1, \cdots, X_n$ are identically distributed, having the common *cdf* $F(x)$, then

$$E_\theta\{T_k(X_1, \cdots, X_n)\} = \frac{(2k+1)!}{k!\,k!} \int_0^1 x[F(x)]^k[1-F(x)]^k \, dF(x)$$

$$= g_k(\theta), \text{ (say)} \tag{2.8}$$

where $g_k(\theta)$ is the expectation of the sample median in a sample of size $(2k+1)$. If the parent *cdf* $F(x)$ be symmetrical about the location parameter $\mu$, then it follows from Theorem 2.2, that $g_k(\theta) = \mu$ for all $k$.

To use the notations of Hoeffding (1948), we now let

$$\psi_{k.c}(x_{\alpha_1}, \cdots, x_{\alpha_c}) = E\{\phi_k(x_{\alpha_1}, \cdots, x_{\alpha_c}, X_{\alpha_{c+1}}, \cdots, X_{\alpha_{2k+1}})\} - g_k(\theta)$$

and

$$\zeta_{k.c} = E\{\psi^2_{k.c}(X_{\alpha_1}, \cdots, X_{\alpha_c})\} \text{ for } c = 0, \cdots, 2k+1.$$

Then, assuming the *cdf* $F(x)$ to possess a finite $\delta$-th order moment for some $\delta \geqslant 2/(k+1)$, it follows from Theorem 2.1, that $\zeta_{k.2k+1} < \infty$, and hence following Hoeffding (1948), we get [using (2.3)]

$$V\{T_k(X_1, \cdots, X_n)\} = \binom{n}{2k+1}^{-1} \sum_{c=1}^{2k+1} \binom{2k+1}{c} \binom{n-2k-1}{2k+1-c} \zeta_{.kc}$$

$$= \frac{(2k+1)^2}{n} \zeta_{k\cdot 1} + 0\,(n^{-2})$$

$$\leqslant \left(\frac{2k+1}{n}\right) \zeta_{k\cdot 2k+1}. \tag{2.9}$$

## 3. Asymptotic Normality of $\{T_k\}$

It now follows from Hoeffding's (1948) well-known theorem on the asymptotic normality of U-statistic, and from (2.8) and (2.9) that for all *cdf*'s possessing a finite $\delta$-th order moment for some $\delta \geqslant 2/(k+1)$, the variable

$$\frac{n^{\frac{1}{2}}\{T_k(X_1, \cdots, X_n) - g_k(\theta)\}}{(2k+1)} \zeta^{\frac{1}{2}}_{k\cdot 1} \tag{3.1}$$

has asymptotically a normal distribution with zero mean and unit variance. It also follows from Theorem 2.2 that $g_k(\theta)$ reduces to the population median for all *cdf*'s, symmetric about $\mu$. But the quantity $\zeta_{k\cdot 1}$ depends appreciably on the parent *cdf*, even when the *cdf* is symmetrical about $\mu$. A knowledge of the value of $\zeta_{k.1}$ or any consistent estimate of it is essential for attaching confidence limits to $g_k(\theta)$ based upon $T_k$ or for testing any hypothesis regarding the numerical value of $\zeta_k(\theta)$. The following is a distribution-free and consistent estimate of $i_{k.1}$.

Let

$$V_{k.j} = \binom{n-1}{2k}^{-1} \sum_{C'_{kj}} \phi_k(X_j, X_{a_1}, \cdots, X_{a_{2k}}) \tag{3.2}$$

where the summation $C_{kj}'$ extends over all $1 \leqslant a_1 < \cdots < a_{2k} \leqslant n$, with $a_i \neq j$ for $i, = 1 \cdots, 2k$. Then, we have

$$T_k(X_1, \cdots, X_n) = n^{-1} \Sigma^n_{j=1} V_{k.j} \tag{3.3}$$

Also let

$$s_k^2 = \frac{1}{n-1} \sum_{j=1}^{n} \{V_{k.j} - T_k\}^2 \tag{3.4}$$

It then follows from a theorem on the structural convergence of Hoeffding's U-statistic, proved by the author (Sen, 1960) that $s_k^2$ is a distribution-free and consistent estimate of $\zeta_{k,1}$. Now, if in (3.2), we replace $X_j$ by $X_{(j)}$ the $j$-th ordered variable in the sample, and denote the corresponding value of $V_{kj}$ by $V_{k(j)}$, for all $j = 1, \cdots, n$; it is then obvious that

$$s_k^2 = \frac{1}{n-1}\left\{\sum_{j=1}^{n} [V_{k(j)} - T_k]^2\right\}$$

$$= \frac{1}{n-1}\left[\sum_{j=1}^{n} V^2_{k(j)} - nT_k^2\right]. \qquad (3\cdot5)$$

And, it follows by simple arguments that

$$V_{k(j)} = \binom{n-1}{2k}^{-1}\left\{\binom{j-1}{k}\binom{n-j}{k} X_{(j)} + \sum_{i=k+1}^{j-1} \binom{i-1}{k}\binom{n-i-1}{k} X_{(i)} + \sum_{i=j+1}^{n-k} \binom{i-2}{k-1}\binom{n-i}{k} X_{(i)}\right\}$$

for $j = 1, \cdots, n$.

Thus, the values of $V_{k(j)}$ can be computed from any given sample, and whence $s_k^2$ is evaluated. In the particular case of $k = 0$, it readily follows that

$$V_{k(j)} = X_{(j)} \quad \text{for all} \quad j = 1, \cdots, n;$$

and hence $s_0^2$ reduces to the sample variance itself.

Once $s_k^2$ has been obtained, we can use

$$t = \frac{n^{\frac{1}{2}} \{T_k(X_1, \cdots, X_n) - g_k(\theta)\}}{(2k+1)\, s_k}$$

for large samples as a normal variate with zero mean and unit variance, and the same may be employed for the purpose of attaching confidence limits to $g_k(\theta)$, or for performing any tests of significance regarding the numerical value of $g_k(\theta)$.

## 4. Stochastic Convergence of $\{T_k\}$

It now follows from the asymptotic normality of the expression in (3.1) that if we select any arbitrarily small $\{\epsilon_n\}$ with $\lim_{n=\infty} \epsilon_n = 0$ but $\lim_{n=\infty} n^{\frac{1}{2}} \epsilon_n = \infty$, then

$$\lim_{n=\infty} P\{g_k(\theta) - \epsilon_n \leqslant T_k(x_1, \cdots, X_n) > g_k(\theta) + \epsilon_n\} = 1 \quad (4 \cdot 1)$$

Whence, the consistency of $T_k$ as an estimate of $g_k(\theta)$.

The sample mean possesses some further stochastic convergence properties. In fact if $X_1, \cdots, X_n$ are identically and independently distributed and have a finite first-order moment, then it follows from Kintchine's Law of Large Numbers that the sample mean converges with probability one to the population mean. The same property is also possessed by $T_k$, even under more relaxed conditions.

THEOREM 4.1. If $X_1, \cdots, X_n$ are identically and independently distributed and the *cdf* $F(x)$ possesses a finite $\delta$-th order moment for some $\delta$:

$$\delta = 1 \text{ if } k = 0,$$

$$> \frac{(4k+1)}{\{(2k+1)(k+1)\}} \quad \text{if} \quad k \geqslant 1,$$

then $T_k$ converges *with probability one* to $g_k(\theta)$.

If further, $\delta > (4k)/\{(2k+1)(k+1)\}$ for $k \geqslant 1$, and $\delta = 1$ if $k = 0$, then $T_k$ converges *in probability one* to $g_k(\theta)$.

*Proof.*—We consider here only the case $k \geqslant 1$; as the case $k = 0$, follows directly from Kintchine's Law of Large Numbers.

It then follows from a theorem on the strong convergence of U-statistic, considered earlier by the author (Sen, 1960) that

$$U(X_1, \cdots, X_n) = \binom{n}{m}^{-1} \sum_{C_m} f(X_{a_1}, \cdots, X_{a_m}) \quad (4.2)$$

(where the summation $C_m$ extends over all possible $1 \leqslant a_1 < \cdots < a_m \leqslant n$) converges with probability one to its expectation, provided

$$E_\theta\{|f(X_{a_1}, \cdots, X_{a_m})|^{1+\delta_0}\} < \infty \text{ for some } \delta_0 > 1 - 1/m,$$

and $U(X_1, \cdots, X_n)$ converges in probability one to its expectation provided, $\delta_0 > 1 - 2/m$.

Thus from (2.2) and (2.3) we get that $T_k(X_1, \cdots, X_n)$ converges with probability one to its expectation, $g_k(\theta)$ provided

$$E\{|\phi_k(X_{\alpha_1}, \cdots, X_{\alpha_{2k+1}})|^{1+\delta_0}\} < \infty \quad \text{for some } \delta_0 > 2k/(2k+1) \tag{4.4}$$

Also, from Theorem 2.1, we note that (4.4) would hold true, provided the parent $cdf\, F(x)$ possesses a finite $\delta$-th order moment, for some $\delta$

$$\delta > \frac{(1+\delta_0)}{(k+1)} > \frac{(4k+1)}{\{(2k+1)(k+1)\}}.$$

Similarly, if we replace the condition in (4.4) by $\delta_0 > (2k-1)/(2k+1)$ we arrive at the convergence of $T_k(X_1, \cdots, X_n)$ to $g_k(\theta)$, in probability one, provided $\delta > 4k/\{(2k+1)(k+1)\}$.

Hence, the theorem.

## 5. Efficiency of $\{T_k\}$

In view of the asymptotic normality of $n^{\frac{1}{2}}\{T_k - g_k(\theta)\}$ a measure of its efficiency may be based upon its variance, and it then follows from (2.9) that for any $cdf\, F(x)$ possessing a finite second-order monent the efficiency of $T_k$ with respect to the sample mean is equal to

$$E_k = \frac{n}{V\{T_k(X_1, \cdots, X_n)\}} \geqslant \frac{1}{\{(2k+1)\zeta_{k.2k+1}\}}. \tag{5.1}$$

In large samples, $E_k$ reduces to $1/\{(2k+1)^2 \zeta_{k.1}\}$. Now $\zeta_{k.1}$ as well as $\zeta_{k.2k+1}$ depends upon the parent $cdf\, F(x)$ and hence, no statement may be made regarding the numerical value of $E_k$ in the general case. By way of illustration, we have considered here the normal distribution and have deduced the values of $E_k$ for $k \leqslant 4$, $n \leqslant 10$.

One advantage of $T_k$ is that it is usable in the censored case, where the smallest $r_1$ and the largest $r_2$ sample observations are censored, for all $r_1, r_2 \leqslant k$. In such a case, the best (minimum variance) linear estimate of the population location based on the sample ordered observations, have been proposed by many workers, and in the particular case of the normal *cdf*, explicit solutions are due to Sarhan and Greenberg (1956). A simplified estimate of the mean has also been proposed by Winsor to Dixon (1960), and this is almost fully efficient for all samples of size less than 10. Here, we have compared the efficiency $T_k$ with respect to the sample mean as well as to the Winsorian estimate of the mean from censored sample with $r_1 = r_2 = k$. (Winsorian estimate has been designated as $\theta_k$*.) For normal *cdf*, $T_k$ is slightly less efficient than $\theta_k$*, though for many non-normal *cdf*'s (*cf.*, Cauchy distribution, Laplace distribution, etc.). $T_k$ is much more efficient than $\theta_k$*.

TABLE I

*The variance of* $T_k$ $(X_1, \cdots, X_n)$ *from a normal population with unit variance**

| Sample size $n$ | Variance of $T_k(X_1, \cdots, X_n)$ | | | | |
|---|---|---|---|---|---|
| | $k=0$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ |
| 1 | 1·0000 | .. | .. | .. | .. |
| 2 | 0·5000 | .. | .. | .. | .. |
| 3 | 0·3333 | 0·4487 | .. | .. | .. |
| 4 | 0·2500 | 0·2982 | .. | .. | .. |
| 5 | 0·2000 | 0·2290 | 0·2868 | .. | .. |
| 6 | 0·1667 | 0·1870 | 0·2147 | .. | .. |
| 7 | 0·1429 | 0·1584 | 0·1768 | 0·2104 | .. |
| 8 | 0·1250 | 0·1375 | 0·1513 | 0·1682 | .. |
| 9 | 0·1111 | 0·1216 | 0·1326 | 0·1442 | 0·1661 |
| 10 | 0·1000 | 0·1089 | 0·1175 | 0·1266 | 0·1383 |

* Tables for the variance-covariance of order statistics from normal distribution, by Sarhan and Greenberg (1956) has been used for the above computations.

As $n \rightarrow \infty$, the efficiency of $T_1$ with respect to the sample mean tends to 94·4%, and that of $T_2$ with respect to the sample mean tends to 87·4%. For normal population, $T_k$ is more efficient than the sample median, while for double exponential distribution, $T_k$ is more efficient than the sample mean. In particular, for Cauchy distribution, the sample mean is absolutely unsuitable, while as $T_k$ for $k \geqslant 1$, appears to be consistent as an estimate of the population location, and for $k \geqslant 2$, $V(T_k)$ exists and converges to zero as $n \rightarrow \infty$,

TABLE II

*The efficiency of* $T_k$ *with respect to* $\theta_k^*$ *and the sample mean for normal parent* cdf *and for* $n \leqslant 10$

| Sample size $n$ | Efficiency of $T_k$ with respect to (in %) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $k = 1$ | | $k = 2$ | | $k = 3$ | | $k = 4$ | |
| | Sample mean | $\theta_1^*$ | Sample mean | $\theta_2^*$ | Sample mean | $\theta_3^*$ | Sample mean | $\theta_4^*$ |
| 3 | 74·3 | 100 | .. | .. | .. | .. | .. | .. |
| 4 | 83·8 | 100 | .. | .. | .. | .. | .. | .. |
| 5 | 87·3 | 98·6 | 69·7 | 100 | .. | .. | .. | .. |
| 6 | 89·1 | 97·6 | 77·6 | 100 | .. | .. | .. | .. |
| 7 | 90·2 | 96·9 | 80·8 | 98·0 | 67·8 | 100 | .. | .. |
| 8 | 90·9 | 96·4 | 82·6 | 96·3 | 74·3 | 100 | .. | .. |
| 9 | 91·4 | 96·1 | 83·8 | 95·1 | 77·1 | 97·8 | 66·9 | 100 |
| 10 | 91·8 | 95·8 | 85·1 | 94·7 | 79·0 | 96·2 | 72·3 | 100 |

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